even more approxmately, in terms of the normal distribution. It is the purpose of this comment to point out that Monte-Carlo methods can be easily used to compute directly the distribution to any desired degree of accuracy. The advantage of such an approach is in requiring a minimum of statistical theory to produce a precise result in an extremely short time.

2. Error Sources

As in the reference, it will be assumed that the parameters β, r, v are normally distributed with covariance matrix Σ. This matrix is obtained by a straightforward linear transformation of the covariance matrix of the fundamental error sources. Although it is convenient if this matrix is diagonal (independent error sources), this is by no means essential.

In the Monte-Carlo procedure it is necessary to generate (on a digital computer) random samples of β, r, v according to the desired normal distribution. This can be done either by generating a random vector of the fundamental error sources (possibly correlated) and then applying the linear transformation, or by directly generating the correlated samples of β, r, v. In both cases it is necessary to generate samples of correlated, normally distributed random variables.

3. Generation of Samples of Correlated Gaussian Variables†

There are well established computer methods for the generation of a vector x of independent Gaussian variables with zero mean and unit variance. It is desired to obtain the vector y with covariance matrix Σ_y by a linear transformation C on x:

$$y = Cx + \bar{y} \tag{2}$$

where $\bar{y}$ is the mean of y and

$$\Sigma_y = E[(y - \bar{y})(y - \bar{y})^T] = CC^T \tag{3}$$

where T indicates the transpose. If C is chosen as a lower triangular matrix, the elements $c_{ij}(j \leq i)$ may be computed sequentially from the relations

$$\sum_{k=1}^{j} c_{ik}c_{jk} = \sigma_{ij} \qquad k \leq j \leq i \tag{4}$$

for $j = 1$, Eqs. (4) have only one term yielding

$$c_{11} = \sigma_{11}^{1/2} \qquad c_{k1} = \sigma_{k1}/c_{11} \tag{5}$$

For $j = 2$, there are two terms, but only one new unknown element is involved. Hence,

$$c_{22} = (\sigma_{22} - c_{21}^2)^{1/2} \qquad c_{k2} = (\sigma_{k2} - c_{k1}c_{21})/c_{22} \tag{6}$$

The pattern continues for higher j, with only one new element appearing in each equation.

With the elements of C known, the correlated components of the random vector y are computed directly in terms of the independent components of x from (2).

4. The Monte-Carlo Simulation

The calculation, then, consists in generating samples of the normally distributed independent components of x (in this case a three-vector), computing the components β, r, v of y from (2), and then computing e^2 from (1).

The time required for such a computation on an IBM 7090 would be less than 1 min for 10,000 runs, including the sorting of the results necessary to produce the complete distribution. An experienced programmer should be able to write and check out such a program in about one week.

In comparison, the method proposed by Beckwith[1] requires the computation of the eigenvalues of a large ($n = 20$ to 30) matrix, or else the assumption that e^2 is normal. In both cases the accuracy of the approximations is difficult to evaluate and requires considerably more statistical sophistication than the direct approach afforded by the Monte-Carlo method.

It should be noted that Skidmore and Braham[2] have presented an alternate approach to the same problem, namely numerical integration of the trivariate Gaussian distribution of β, r, v over the region in β, r, v space bounded by a particular value of e^2. This procedure is not difficult to apply in the event that a multivariate Gaussian integration program is available. This latter reference also contains a relatively complete description of the general problem of statistical analysis of satellite trajectories

† The methods of this section were pointed out to the author by L. J. Skidmore.

References

[1] Beckwith, R. E., "Approximate distribution of nearly circular orbits," AIAA J. 2, 913–916 (1964).

[2] Braham, H. S. and Skidmore, L. J., "Guidance error analysis of satellite trajectories," J. Aerospace Sci. **29**, 1091–1101 (1962).

Reply by Author to D. A. Conrad

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1. Introduction

The preceding comment advocates the superiority of Monte Carlo methods over the methods presented in Ref. 1 for answering certain questions pertaining to the precision of "nearly circular" orbits. Without agreeing or disagreeing with D. A. Conrad, the present note seeks to place the issue in clearer perspective.

2. Monte Carlo Method

A rich, if diffuse, literature exists in *model sampling* or *Monte Carlo method*, as the technique is now more popularly known. Principal applications have been made in the areas of nuclear studies, logistics, heuristic problem solving, and evaluation of complicated integrals. Whether the original problem is deterministic or stochastic, the Monte Carlo method consists of building and "playing" an appropriate game of chance in which stochastic convergence of relevant sample statistics to basic system parameters is assured under very general conditions by the strong law of large numbers and the central limit theorem and in which distribution laws of arbitrary statistics can be determined to any desired degree of accuracy by appropriate techniques, such as that due to Kolmogorov-Smirnov.

The efficient use of Monte Carlo techniques depends heavily upon one's ability to transform one game of chance into another one, in which the expected values of the important system statistics remain unchanged, but for which the statistics in the transformed game have smaller sampling variances than those of the original game.

The writer, who has used the Monte Carlo method to treat a great variety of problems which posed particular analytic difficulty, has no quarrel with others who would employ it for the same reason.† He would point out, however, that many questions that can be answered by a straightforward sensitivity analysis of an analytic model become costly exercises

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† Today's rather widespread practice of usage of Monte Carlo or gaming techniques by persons who are unprepared either to recognize analytically solvable problems or treat them by analytic methods, or to comprehend the intricacies to modern statistical theory, a commentary not relevant to this reply to the preceding note, is to be deplored, however.

when approached by sampling methods. On the other hand, and in all fairness, it should be admitted that the problem of determining the distribution of general orbits (a topic of importance in assessing the probability of "hitting" a specific "window" in position-velocity space) probably can be handled best by Monte Carlo methods. A good rule to follow is to allow the intrinsic characteristics of the particular problem to dictate the method of treatment to be employed.‡

3. Some Points of Confusion

Contrary to a statement in the preceding comment, it is not assumed in Ref. 1 that β, r, and v are normally distributed with covariance matrix Σ; but it can be inferred, from assumptions made regarding the fundamental guidance parameters, that these quantities possess, approximately, a trivariate normal distribution, with covariance matrix $A\Sigma_x A'$, where the prime denotes the transpose of a matrix. Here, $A = (a_{ij})$ is the $3 \times n$ matrix associated with Eqs. (3) and (4) of Ref. 1, and Σ_x is the covariance matrix of the "errors" associated with the n fundamental guidance parameters.

The x of the preceding note is a random vector of three components, which are distributed independently and identically according to the standard normal law; it is not to be confused with the x of Ref. 1.

4. Monte Carlo vs Analytic Treatment

Two ways of employing the Monte Carlo method in this situation suggest themselves. One is to generate the random n-tuple $(X_1, \ldots, X_n)$ of "errors" in the fundamental guidance parameters, and through Eqs. (3), or equivalently, through Eqs. (6) and (5) of Ref. 1 to simulate (1), the square of the orbital eccentricity. [In this connection the writer is grieved to call attention to the absence of a sign of equality between the f term and the $\sin^2$ term in formula (1) of Ref. 1.] The other method, favored in the preceding note and possibly more efficient from the sampling viewpoint, is to draw random triples from the trivariate normal population with means zero and covariance matrix $A\Sigma_x A'$.

Either method necessitates the computation of the elements of A and Σ_x, which is possibly the most formidable task in the exercise.

The analytic procedure proposed in Ref. 1 was not represented as being superior to any other legitimate method. The computation of the eigenvalues of matrix $(c_{ij}\sigma_i\sigma_j)$, as advocated in Ref. 1, may not be a particularly difficult task, however. The issue depends on the rank of this matrix; if it is $r < n$, then it is well known that $n - r$ of the eigenvalues are zero. Numerical methods for computing the eigenvalues of a real, symmetric matrix are well known and generally available (cf. Ref. 2). Routines for calculating the eignenvalues of a positive-definite (or positive-semidefinite) matrix according to decreasing numerical magnitude are in common use in *factor analysis* studies; they would appear to be especially suitable in the present application where eigenvalues smaller than a preassigned positive constant could be ignored, thus simplifying the analytic treatment by a certain amount.

5. Generation of Random Numbers

The validity of Monte Carlo applications, in a precise mathematical sense, depends on the intrinsic quality of the "random" number sequences employed. The sequences with which we are compelled, for reasons of economy, to work are not random, but "pseudo-random," in the sense that they possess at least some of the properties of random sequences. The particular properties demanded in different applications vary.

A fundamental and commonly required property, moreover, one which is essential in the application recommended in the preceding note, is that the sequence of numbers be *equidistributed,* as defined in Ref. 3. The currently favored mixed congruential methods for generating pseudo-random sequences[4] are the computer implementation of what, in full-precision arithmetic, are called *multiply sequences* in Ref. 3.

Theorem 20 in Ref. 3 establishes that r-dimensional derived sequences (obtained from a multiply sequence) cannot be equidistributed in the r-dimensional unit cube, for any $r > 1$. This result, valid for full-precision arithmetic, casts grave doubts on the implicit postulate that computer-generated sequences possess this desirable property, without which random ordered r-tuples cannot be produced.

The effect of the foregoing is that computed quantities that depend on these r-tuples, such as orbital eccentricity or its square, cannot be considered to be random observations. Consequently, one really has little a priori grounds for believing in the validity of standard statistical procedures when applied to Monte Carlo experiments and no obvious direct method for testing for such validity. As an example, presumed 90% confidence limits for an important system parameter may in fact have an associated true confidence coefficient considerably smaller than 0.9.

‡ The writer, in fact, advocated and employed the Monte Carlo method in treating the original version of the problem of Ref. 1, but did not consider the application worthy of publication.

References

[1] Beckwith, R. E., "Approximate distribution of nearly circular orbits," AIAA J. 2, 913–916 (1964).

[2] Cooley, W. W. and P. R. Lohnes, *Multivariate Procedures for the Behavioral Sciences* (John Wiley & Sons, Inc., New York, 1962), Chaps. 8 and 9.

[3] Franklin, J. N., "Deterministic simulation of random processes," Math. Computation 17, 28–59 (1963).

[4] Hull, T. E. and Dobell, A. R., "Random number generators," SIAM Rev. 4, 230–254 (1962).

Errata: "Optimal Variable-Thrust Transfer of a Power-Limited Rocket between Neighboring Circular Orbits"

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[AIAA J. 2, 339–343 (1964)]

In the above article, the equations for the out-of-plane motion are incorrect and should be replaced by the following:

$$\frac{C_5}{C_3} = \frac{\tan\tau_f + (w_f/z_f)}{1 - (w_f/z_f)\tan\tau_f} \tag{31}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)z_f - (\tau_f\sin\tau_f)w_f}{\tau_f^2 - \sin^2\tau_f} \tag{40}$$

$$A_z = 2C_3\left\{\left[\frac{\tan\tau_f + (w_f/z_f)}{1 - (w_f/z_f)\tan\tau_f}\right]\cos\tau - \sin\tau\right\} \tag{44}$$

$$\frac{J}{w_0^3 r_0^2} = \frac{(y_f/r_0)^2(5\tau_f + 3\sin\tau_f)}{8[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1 - \cos\tau_f)]} + \frac{i^2}{\tau_f + |\sin\tau_f|} \tag{45}$$

$$\frac{z_f}{r_0 i} = \left(\frac{1 \pm \cos\tau_f}{2}\right)^{1/2} \tag{49}$$

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